Chemical algebra. VI: G-weighted metrics of non-compact groups: group of translations in the Euclidean space

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The generalized definition equation of a G-weighted metric ds^2 from the datum of any group G acting onto a vector space mapped by a continuous numerical function μ is applied when $E = R^n$ and $G =$ the group of translations in R^n. Here, G does not act linearly in R^n and R^n is considered as an affine space. The solution reads $ds^2 = -d^2(\ln I)/(Bp)$, $I = (4i\pi x^0)^{n/2} \cdot \Psi$, where $x^0 = -i/(2p)$, Ψ is a solution of the Schrödinger-type equation $\Delta\Psi + i\, \partial\Psi/\partial x^0 = 0$, and B is a uniform term depending on x^0. When $n = 3$, p is interpreted as the reciprocal of a time variable. Attempts to identify ds^2 with the spatial part of a space-time metric of general relativity failed except for the flat Robertson and Walker spaces. In the simplest case, $B = 1/R^2(t)$ and $\Psi(p, r) = e^{-pr^2/2}$. A uniform but non-constant "imaginary potential energy" of the space can be formally derived: $\mathrm{V}(x^0) = 3i/(2x^0)$. Despite a striking formal link with tools of physical mathematics, no physical validation of the propositions of chemical algebra is claimed.

1. Introduction

Vector translations in R^n are not linear. Although the theory set out hitherto refers to linear representations [1–4], the definitions of $K_p(\mathbf{u}, \mathbf{v})$ and $\Phi_{\mathbf{u},\mathbf{v}}(x)$ can also be formally applied to any non-linear operation of a group G onto an Euclidean vector space. However, attention has to be paid not to use formula such as $g(\mathbf{u} + \mathbf{v}) = g\mathbf{u} + g\mathbf{v}$ or as $\|g\mathbf{u}\| = \|\mathbf{u}\|$. R^n is also considered as an affine Euclidean space, and the contravariant notation for the components $x^1, \ldots, x^n$ of a vector $\mathbf{u}$ in R^n is adopted.

2. Insights into a generalized equation ($\mathbb{E}$) for a non-compact group G

2.1. FORMAL DERIVATION OF $\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}$ FOR THE GROUP OF TRANSLATIONS IN R^n

The group G of the translations in R^n is not compact and not finite Haar measure is available for G [5]. G is topologically equivalent to R^n itself, and the vector of a translation g in G is denoted by $\mathbf{t}$ in R^n. In the case of linear representations, an operation g acts as a linear application whose components in the canonical basis set are linear forms belonging to the dual space of R^n: by extension to affine applications, it is therefore relevant to adopt a covariant notation for the components $t_1, \ldots, t_n$ of the vector $\mathbf{t}$ defining the translation g. The notation $\int_G \ldots dg$ is used for the current notation of convergent integrals over R^n multiplied by an arbitrary factor $q^{n/2}$ whose dimension is the reciprocal of a volume: if F is an integrable map of $G \approx R^n$ and if $d\tau$ denotes the volume element in R^n:

$$\int_G F(g)\, dg \stackrel{\text{def.}}{=} q^{n/2} \int_{R^n} F(\mathbf{t})\, d\tau = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(t_1, \ldots, t_n) q^{n/2}\, dt_1 \ldots dt_n\,.$$

The occurrence of the symbol $q^{n/2}$ is dictated by the non-availability of a well-defined Haar measure on a non-compact group: q serves the requirement that $dg = q^{n/2} dt_1 \ldots dt_n$ must be adimensional. One has to keep in mind that $q^{n/2}$ tends to some infinite quantity when the limits of the experienced space draw nearer to infinity (the condition $\int_G dg = 1$ might still be then formally satisfied). The notation is *formally* used to justify the final formulation of the equation ($\mathbb{E}$) in the case of a non-compact group. The group of translations in R^n is non-compact, and the local definition of $\Phi_{\mathbf{u},\mathbf{v}}$ for $\mathbf{v} = \mathbf{u} + d\mathbf{u}$ reads [6]

$$\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(\gamma ds) = 1 + pB(\mathbf{u}, d\mathbf{u}) ds^2$$

with

- $ds^2 = (d\sigma/\gamma)^2$ (if γ is formally adimensional, both ds and $d\sigma$ have the dimension of a length)
- $B(\mathbf{u}, d\mathbf{u}) = \iint_{G^2} \mu^2_{\mathbf{u},\mathbf{u}}(g) \mu^2_{\mathbf{u},\mathbf{u}}(k) \dfrac{(g\mathbf{u} - g(\mathbf{u} + d\mathbf{u}) | k\mathbf{u} - k(\mathbf{u} + d\mathbf{u}))}{\|g\mathbf{u} - g(\mathbf{u} + d\mathbf{u})\| \cdot \|k\mathbf{u} - k(\mathbf{u} + d\mathbf{u})\|} dg\, dk \Big/ \left(\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg\right)^4$ (this expression was denoted as $B^2(\mathbf{u}, d\mathbf{u})$ in ref.[6].)

Since the equality $g(\mathbf{u} + d\mathbf{u}) = g\mathbf{u} + gd\mathbf{u}$ is no longer valid for a non-linear representation, the calculation of $B(\mathbf{u}, d\mathbf{u})$ proceeds differently. The integral over G^2 is identified with an integral over $(R^n)^2$: let $\mathbf{g}$ denote the vector of the translation g, and $\mathbf{k}$ the vector of the translation k. Then,

$$\begin{aligned} C_{g,g,k,k}(\mathbf{u}, \mathbf{u} + d\mathbf{u}) &= \frac{(g\mathbf{u} - g(\mathbf{u} + d\mathbf{u}) | k\mathbf{u} - k(\mathbf{u} + d\mathbf{u}))}{\|g\mathbf{u} - g(\mathbf{u} + d\mathbf{u})\| \cdot \|k\mathbf{u} - k(\mathbf{u} + d\mathbf{u})\|} \\ &= \frac{(\mathbf{g} + \mathbf{u} - \mathbf{g} - \mathbf{u} - d\mathbf{u} | \mathbf{k} + \mathbf{u} - \mathbf{k} - \mathbf{u} - d\mathbf{u})}{\|\mathbf{g} + \mathbf{u} - \mathbf{g} - \mathbf{u} - d\mathbf{u}\| \cdot \|\mathbf{k} + \mathbf{u} - \mathbf{k} - \mathbf{u} - d\mathbf{u}\|} \\ &= \frac{\|d\mathbf{u}\|^2}{\|d\mathbf{u}\|^2} = 1\,. \end{aligned}$$

Thus,

$$B(\mathbf{u}, d\mathbf{u}) = \int\int_{G^2} \mu^2_{\mathbf{u},\mathbf{u}}(g)\mu^2_{\mathbf{u},\mathbf{u}}(k) \cdot 1 \cdot dg\, dk \Big/ \left(\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg\right)^4,$$

$$B(\mathbf{u}, d\mathbf{u}) = \left(\int_G \mu^2_{\mathbf{u},\mathbf{u}}(g)\, dg\right)^2 \Big/ \left(\int_G \mu_{\mathbf{u},\mathbf{u}}(g)\, dg\right)^4.$$

Using the definition of the symbol $dg = q^{n/2}dt_1 \dots dt_n$ we get

$$B(\mathbf{u}, d\mathbf{u}) = \left(\int_{R^n} \mu^2_{\mathbf{u},\mathbf{u}}(g)q^{n/2}\, dt_1 \dots dt_n\right)^2 \Big/ \left(\int_{R^n} \mu_{\mathbf{u},\mathbf{u}}(g)q^{n/2}\, dt_1 \dots dt_n\right)^4,$$

$$B(\mathbf{u}, d\mathbf{u}) = q^{-n}\left(\int_{R^n} \mu^2_{\mathbf{u},\mathbf{u}}(g)\, dt_1 \dots dt_n\right)^2 \Big/ \left(\int_{R^n} \mu_{\mathbf{u},\mathbf{u}}(g)\, dt_1 \dots dt_n\right)^4.$$

Since $q \approx \infty$, B can remain finite if

$$b(\mathbf{u}, d\mathbf{u}) = \left(\int_{R^n} \mu^2_{\mathbf{u},\mathbf{u}}(g)\, dt_1 \dots dt_n\right)^2 \Big/ \left(\int_{R^n} \mu_{\mathbf{u},\mathbf{u}}(g)\, dt_1 \dots dt_n\right)^4$$

is infinite too.

The equation is written as

$$\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(d\sigma) = 1 + pB(\mathbf{u}, d\mathbf{u})ds^2.$$

$B(\mathbf{u}, d\mathbf{u})$ does not depend on $d\mathbf{u}$, and it will be seen that the standard hypothesis on $\mu_{\mathbf{u},\mathbf{u}}$ entails that $B(\mathbf{u}, d\mathbf{u})$ does not depend on $\mathbf{u}$ either. The left-hand side of eq. ($\mathbb{E}$) reduces to

$$\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(\gamma ds) = 1 + pB(\mathbf{u})ds^2.$$

In conclusion, after calculation of the local pairing product $K_p(\mathbf{u}, \mathbf{u} + d\mathbf{u})$, the equation "$\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(d\sigma) = K_p(\mathbf{u}, \mathbf{u} + d\mathbf{u})$" is expected to define a classical Riemannian metric in the selected coordinate system.

2.2. FORMULATION OF PAIRING PRODUCTS FOR THE GROUP OF TRANSLATIONS IN R^n

Although no conditions are precised, the definition of K_p and eq. ($\mathbb{E}$) are formally applied to the non-compact group of translations in $E = R^n$. For the sake of brevity, let us define the two-variable map μ on $(R^n)^2$: $\mu(\mathbf{u}, \mathbf{t}) = \mu_{\mathbf{u},\mathbf{u}}(g)$, where $\mathbf{t}$ denotes the vector of a translation g.

$$K^p_p(\mathbf{u}, \mathbf{u} + d\mathbf{u})$$
$$= \frac{\int \mu(\mathbf{u},\mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} + \mathbf{u} - \mathbf{u}\|^2\right] d\tau \cdot \int \mu(\mathbf{u},\mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} + \mathbf{u} + d\mathbf{u} - \mathbf{u} - d\mathbf{u}\|^2\right] d\tau}{\int \mu(\mathbf{u},\mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} + \mathbf{u} + d\mathbf{u} - \mathbf{u}\|^2\right] d\tau \cdot \int \mu(\mathbf{u},\mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} + \mathbf{u} - \mathbf{u} - d\mathbf{u}\|^2\right] d\tau},$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ (covariant vector), $d\tau = dt_1 \ldots dt_n$, and where the integral symbol $\int$ stretches from $-\infty$ to $+\infty$ for all the arguments $t_1, \ldots, t_n$.

$$K_p^p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) = \frac{\left\{\int \mu(\mathbf{u}, \mathbf{t} \exp\left[-\frac{p}{2}\|\mathbf{t}\|^2\right] d\tau\right\}^2}{\int \mu(\mathbf{u}, \mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} + d\mathbf{u}\|^2\right] d\tau \cdot \int \mu(\mathbf{u}, \mathbf{t}) \exp\left[-\frac{p}{2}\|\mathbf{t} - d\mathbf{u}\|^2\right] d\tau}.$$

A second-order Taylor expansion in $d\mathbf{u}$ yields

$$K_p^p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) \approx \left\{\int \mu(\mathbf{u}, \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2}\, d\tau\right\}^2 \Big/$$

$$\left(\int \mu(\mathbf{u}, \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2} \left[1 - \frac{p}{2}\|d\mathbf{u}\|^2 - p(\mathbf{t}|d\mathbf{u}) + \frac{p^2}{2}(\mathbf{t}|d\mathbf{u})^2\right] d\tau \right.$$
$$\left. \times \int \mu(\mathbf{u}, \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2} \left[1 - \frac{p}{2}\|d\mathbf{u}\|^2 + p(\mathbf{t}|d\mathbf{u}) + \frac{p^2}{2}(\mathbf{t}|d\mathbf{u})^2\right] d\tau \right).$$

Let us define the integrals:

$$I = \int \mu(\mathbf{u}, \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2}\, d\tau; \quad J = \int \mu(\mathbf{u}, \mathbf{t}) \left(\mathbf{t} \middle| \frac{d\mathbf{u}}{\|d\mathbf{u}\|}\right) e^{-p\|\mathbf{t}\|^2/2}\, d\tau;$$

$$K = \int \mu(\mathbf{u}, \mathbf{t}) \left(\mathbf{t} \middle| \frac{d\mathbf{u}}{\|d\mathbf{u}\|}\right)^2 e^{-p\|\mathbf{t}\|^2/2}\, d\tau.$$

Then,

$$K_p^p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) \approx \frac{I^2}{\left\{I\left[1 - \frac{p}{2}\|d\mathbf{u}\|^2\right] - pJ\|d\mathbf{u}\| + \frac{p^2}{2}K\|d\mathbf{u}\|^2\right\}\left\{I\left[1 - \frac{p}{2}\|d\mathbf{u}\|^2\right] + pJ\|d\mathbf{u}\| + \frac{p^2}{2}K\|d\mathbf{u}\|^2\right\}},$$

$$K_p^p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) \approx \frac{1}{\left\{1 - \frac{p}{2}\|d\mathbf{u}\|^2 - p\frac{J}{I}\|d\mathbf{u}\| + \frac{p^2}{2}\frac{K}{I}\|d\mathbf{u}\|^2\right\}\left\{1 - \frac{p}{2}\|d\mathbf{u}\|^2 + p\frac{J}{I}\|d\mathbf{u}\| + \frac{p^2}{2}\frac{K}{I}\|d\mathbf{u}\|^2\right\}},$$

$$K_p^p(\mathbf{u}, \mathbf{u} + d\mathbf{u}) \approx \frac{1}{1 - \frac{p}{2}\|d\mathbf{u}\|^2 - p\frac{J}{I}\|d\mathbf{u}\| + \frac{p^2}{2}\frac{K}{I}\|d\mathbf{u}\|^2 - \frac{p}{2}\|d\mathbf{u}\|^2 + p\frac{J}{I}\|d\mathbf{u}\| + \frac{p^2}{2}\frac{K}{I}\|d\mathbf{u}\|^2 - p^2\left(\frac{J}{I}\right)^2\|d\mathbf{u}\|^2},$$

$$K_p^p(\mathbf{u}, \mathbf{u}+d\mathbf{u}) \approx \frac{1}{1-p\left[1-p\frac{K}{I}+p\left(\frac{J}{I}\right)^2\right]\|d\mathbf{u}\|^2}.$$

And finally,

$$K_p^p(\mathbf{u}, \mathbf{u}+d\mathbf{u}) \approx 1+p\left[1-p\frac{K}{I}+p\left(\frac{J}{I}\right)^2\right]\|d\mathbf{u}\|^2.$$

2.3. *G*-WEIGHTED METRICS OF THE GROUP OF TRANSLATIONS IN R^n

From the preceding sections, the definition equation ($\mathbb{E}$) of $d\sigma^2 = (\gamma ds)^2$ is written down by equating $K_p^p(\mathbf{u}, \mathbf{u}+d\mathbf{u})$ with $\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(\gamma ds) = 1 + pq^{-n}b(\mathbf{u})ds^2$:

$$ds^2 = \frac{1}{B(\mathbf{u})}\left[1-p\frac{K}{I}+p\left(\frac{J}{I}\right)^2\right]\|d\mathbf{u}\|^2.$$

Since $\mathbf{u} = (x^1, \ldots, x^n)$, $(\mathbf{t}|d\mathbf{u}) = \sum t_i dx^i$, $d\tau = dt_1 \ldots dt_n$, $\|d\mathbf{u}\|^2 = \sum (dx^i)^2$, let

$$J_i = \int \mu(\mathbf{u}, \mathbf{t}) t_i e^{-p\|\mathbf{t}\|^2/2}\, d\tau,$$

$$K_i = \int \mu(\mathbf{u}, \mathbf{t}) t_i^2 e^{-p\|\mathbf{t}\|^2/2}\, d\tau,$$

$$L_{ij} = \int \mu(\mathbf{u}, \mathbf{t}) t_i t_j e^{-p\|\mathbf{t}\|^2/2}\, d\tau \quad (\text{with } L_{ii} = K_i).$$

Then,

$$(J\|d\mathbf{u}\|)^2 = \left(\sum J_i dx^i\right)^2 = \sum_{1\leqslant i\leqslant n}\ \sum_{1\leqslant j\leqslant n} J_i J_j dx^i dx^j$$
$$= \sum_{1\leqslant i\leqslant n} J_i^2 (dx^i)^2 + 2 \sum_{1\leqslant i<j\leqslant n} J_i J_j dx^i dx^j.$$

Likewise,

$$K\|d\mathbf{u}\|^2 = \sum_{1\leqslant i\leqslant n} K_i (dx^i)^2 + 2 \sum_{1\leqslant i<j\leqslant n} L_{ij} dx^i dx^j.$$

Thus,

$$B(\mathbf{u})ds^2 = \sum_{i=1}^{n}\left[1-p\frac{K_i}{I}+p\left(\frac{J_i}{I}\right)^2\right](dx^i)^2 + 2p\sum_{1\leqslant i<j\leqslant n}\left(\frac{J_iJ_j}{I^2}-\frac{L_{ij}}{I}\right)dx^i dx^j.$$

This expression is now simplified by using an integration by part in J_i:

$$J_i = \int \mu(\mathbf{u}, t_1, \ldots, t_n) t_i e^{-p(t_1^2+\ldots+t_n^2)/2}\, d\tau = -\frac{1}{p}\int e^{-p(\|\mathbf{t}\|^2-t_i^2)/2}$$

$$\times \left\{ \int_{-\infty}^{+\infty} \mu(\mathbf{u}, t_1, \ldots, t_n)(-pt_i) e^{-pt_i^2/2}\, dt_i \right\} \frac{d\tau}{dt_i}\,,$$

$$J_i = \frac{-1}{p}\int e^{-p(\|\mathbf{t}\|^2-t_i^2)/2} \Big\{ [\mu(\mathbf{u}, t_1, \ldots, t_n) e^{-pt_i^2/2}]_{t_i=-\infty}^{t_i=+\infty}$$

$$- \int_{t_i=-\infty}^{t_i=+\infty} \frac{\partial \mu}{\partial t_i}(\mathbf{u}, t_1, \ldots, t_n) e^{-pt_i^2/2} dt_i \Big\} \frac{d\tau}{dt_i}\,.$$

If we assume $\mu(\mathbf{u}, t_1, \ldots, t_n) e^{-pt_i^2/2} \underset{t_i \to \infty}{\longrightarrow} 0$, then,

$$J_i = \frac{1}{p}\int \frac{\partial \mu}{\partial t_i}(\mathbf{u}, t_1, \ldots, t_n) e^{-p\|\mathbf{t}\|^2/2}\, d\tau\,.$$

Likewise, it is easily shown that under the same condition, if $i \neq j$,

$$L_{ij} = \frac{1}{p^2}\int \frac{\partial^2 \mu}{\partial t_i \partial t_j}(\mathbf{u}, t_1, \ldots, t_n) e^{-p\|\mathbf{t}\|^2/2}\, d\tau\,,$$

and if $i = j$,

$$K_i = L_{ii} = \frac{1}{p}\int \left(\mu + \frac{1}{p}\frac{\partial^2 \mu}{\partial t_i^2} \right)(\mathbf{u}, t_1, \ldots, t_n) e^{-p\|\mathbf{t}\|^2/2}\, d\tau\,.$$

Therefore, the expression of ds^2 becomes homogeneous:

$$B(\mathbf{u}) ds^2$$

$$= \frac{1}{p}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\int \frac{\partial \mu}{\partial t_i} e^{-p\|\mathbf{t}\|^2/2}\, d\tau \int \frac{\partial \mu}{\partial t_j} e^{-p\|\mathbf{t}\|^2/2}\, d\tau}{I^2} - \frac{\int \frac{\partial^2 \mu}{\partial t_i \partial t_j} e^{-p\|\mathbf{t}\|^2/2}\, d\tau}{I} \right) dx^i dx^j\,.$$

Generally speaking, a relevant form of $\mu_{\mathbf{u},\mathbf{v}}(g)$ has been propounded, namely [7]: $\mu_{\mathbf{u},\mathbf{v}}(g) = m(g)\pi(g\mathbf{u})\pi(g\mathbf{v})$. Thus, $\mu_{\mathbf{u},\mathbf{u}}(g) = m(g)\pi^2(g\mathbf{u})$, where m and π are one-variable maps of G and R^n, respectively. Assuming $m(g) = 1$ (all translations are "equally possible"), the function $\mu(\mathbf{u}, \mathbf{t})$ defined on $E^2 = R^{2n}$ is to have the form

$$\mu(\mathbf{u}, \mathbf{t}) = \mu(x^1, \ldots, x^n, t_1, \ldots, t_n) = \pi^2(\mathbf{u} + \mathbf{t}) = \mu(x^1 + t_1, \ldots, x^n + t_n)\,,$$

where $\pi^2(\mathbf{y}) = \mu(\mathbf{y})$ is a now a function of the argument $\mathbf{y} = (y^1, \ldots, y^n) \in E = R^n$.

This assumption entails two consequences:

a) $B(\mathbf{u})$ is a constant:

$$B = q^{-n}\left(\int_{R^n} \mu^2(x^1 + t_1, \ldots, x^n + t_n)\, dt_1 \ldots dt_n\right)^2 \Big/$$
$$\left(\int_{R^n} \mu(x^1 + t_1, \ldots, x^n + t_n)\, dt_1 \ldots dt_n\right)^4 \quad (\text{for } \mathbf{u} = (x^1, \ldots, x^n))$$
$$= q^{-n}\left(\int_{R^n} \mu^2(t_1, \ldots, t_n)\, dt_1 \ldots dt_n\right)^2 \Big/ \left(\int_{R^n} \mu(t_1, \ldots, t_n)\, dt_1 \ldots dt_n\right)^4,$$

which is independent of $\mathbf{u}$.

b) It renders the integrals I, J_i, L_{ij} convolution products. Then,

$$\frac{\partial \mu}{\partial t_i}(x^1, \ldots, x^n, t_1, \ldots, t_n) = \frac{\partial \mu}{\partial y^i}(\mathbf{u} + \mathbf{t}) = \frac{\partial \mu}{\partial x^i}(x^1, \ldots, x^n, t_1, \ldots, t_n)$$

and subsequently

$$J_i = \frac{\partial I}{\partial x^i}; \quad L_{ij} = \frac{\partial^2 I}{\partial x^i \partial x^j}.$$

Thus,

$$Bds^2 = \frac{1}{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\frac{\partial I}{\partial x^i}\frac{\partial I}{\partial x^j}}{I^2} - \frac{\frac{\partial^2 I}{\partial x^i \partial x^j}}{I}\right) dx^i dx^j$$
$$= -\frac{1}{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2(\ln I)}{\partial x^i \partial x^j} dx^i dx^j,$$

where

$$I = \int \mu(\mathbf{u} + \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2}\, d\tau = \int \mu(\mathbf{u} - \mathbf{t}) e^{-p\|\mathbf{t}\|^2/2}\, d\tau.$$

In conclusion, ds^2 is an "exact second differential" defined by

$$\boxed{-Bp = \frac{d^2(\ln I)}{ds^2}}$$

It must be stressed that this definition refers to the given "rectangular" coordinate system initially selected to formulate eq. ($\mathbb{E}$). Although $\ln I$ is supposed to be a scalar tensor, it is known that $\partial^2(\ln I)/\partial x^i \partial x^j$ is neither a (0, 2), a (2, 0) nor a (1, 1) tensor (in contrast, the gradient $\partial(\ln I)/\partial x^i$ is a covariant (0, 1) tensor). The bordered definition has no tensorial character, that is, in another coordinate system $\{x'^i\}$, the linear element

$$d\sigma^2 \neq \frac{\partial x^k}{\partial x'^i}\frac{\partial x^h}{\partial x'^j}\frac{\partial^2(\ln I)}{\partial x^h \partial x^k} dx'^i dx'^j.$$

Remark

This point and the differential formulation of ds^2 naturally prompt us to attempt to replace the ordinary differential of the equation by a covariant derivative in order to formulate an analogous tensorial definition of ds^2 [8]. Indeed, the preceding equation can be written as: $-Bpds^2 = d(\nabla_h(\ln I)dx^h)$, where $\nabla_h(\ln I) = \partial(\ln I)/\partial x^h$ is the gradient of $\ln I$. Since $\nabla_h(\ln I)$ is a covariant tensor, a tensorial differential equivalent is defined through the covariant derivatives of the components $\nabla_h(\ln I)$, $1 \leqslant h \leqslant n$. Thus, a tensorial definition of ds^2 might be given by: $-Bpds^2 = D(\nabla_h(\ln I)dx^h)$, where D denotes the absolute differential of a tensor. However, this attempt is fruitless. Indeed,

$$D(\nabla_h(\ln I)dx^h) = (\nabla_h(\ln I))_{|k}dx^k dx^h\,, \quad \nabla_h(\ln I)_{|k} = \frac{\partial^2(\ln I)}{\partial x^h \partial x^k} + \Gamma_h{}^m{}_k \frac{\partial(\ln I)}{\partial x^m}\,,$$

where $\Gamma_h{}^m{}_k$ denote the Christoffel symbols of the second kind with respect to the symmetric covariant tensor field g_{hk} to be determined:

$$\Gamma_h{}^m{}_k = g^{lm}\Gamma_{hlk}$$

(where g^{lm} is the contravariant reciprocal of $g_{lm} : g_{il}g^{lm} = \delta_i^m$),

$$\Gamma_{hlk} = \frac{1}{2}\left[\frac{\partial g_{lk}}{\partial x^h} + \frac{\partial g_{lh}}{\partial x^k} - \frac{\partial g_{hk}}{\partial x^l}\right] \quad \text{(Christoffel symbols of the first kind)}.$$

Thus, the equation reads $-Bpg_{hk} = (\nabla_h(\ln I))_{|k}$, i.e.

$$-Bpg_{hk} = \frac{\partial^2(\ln I)}{\partial x^h \partial x^k} + \Gamma_h{}^m{}_k \frac{\partial(\ln I)}{\partial x^m}\,,$$

where the g_{hk}'s are unknown for a given function $\ln I$.

Since the affine connection corresponding to g_{ij} is symmetric, the covariant derivative of g_{ij} expressed in terms of this particular connection vanishes identically, i.e.: $g_{hk|i} = 0$ (Ricci's lemma). Thus, if $Bp \neq 0$, $(\nabla_h(\ln I))_{|k|i} = 0$.

On the other hand, the Ricci's identity for a covariant vector field Y_h is written as

$$-K_h{}^l{}_{ki}Y_l - S_k{}^l{}_i Y_{h|l} = Y_{h|k|i} - Y_{h|i|k}\,,$$

where $K_h{}^l{}_{ki}$ denotes the curvature tensor and $S_k{}^l{}_i$ denotes the torsion tensor. Since the Christoffel symbols are symmetric, $S_k{}^l{}_i = 0$, and the Ricci's identity boils down to: $-K_h{}^l{}_{ki}Y_l = Y_{h|k|i} - Y_{h|i|k}$. Let us apply this identity for $Y_h = \nabla_h(\ln I)$:

$$-K_h{}^l{}_{ki}\nabla_l(\ln I) = \nabla_h(\ln I)_{|k|i} - \nabla_h(\ln I)_{|i|k} = 0 - 0 = 0\,.$$

The equality $K_h{}^l{}_{ki}Y_l = 0$ is a necessary and sufficient condition to be satisfied by a *parallel* covariant vector field Y_h on a curved space such that $K_h{}^l{}_{ki}Y_l \neq 0$. Therefore, $\nabla_h(\ln I)$ is a parallel gradient, i.e.: $\nabla_h(\ln I)_{|k} = 0$. Consequently, if we suppose $Bp \neq 0$,

$$g_{hk} = -\frac{1}{bp}(\nabla_h(\ln I))_{|k} = 0 \quad \text{and} \quad ds^2 = 0\,.$$

In conclusion, the sole (non-zero!) metric that might be tensorially represented as an absolute second differential corresponds to a flat space (where $K_h{}^l{}_{ki}\,Y_l = 0$)!

3. Speculations for an interpretation of I and ds^2

3.1. ON THE EXTENSION OF THE EQUATION (𝔼) FOR COMPLEX VALUES OF p AND μ

It appears not straightforward to formulate a natural extension of eq. (𝔼) entailing a (positive or negative) real solution ds^2 for complex-valued functions μ and/or for imaginary parameters $p' = ip, p \in R$ [9]. Therefore, the simplest formal extension of (𝔼) is considered for complex values of μ and $p' = ip$, even though the solution ds^2 is no longer real.

Replacing p by $p' = ip$ and the real function $\mu(\mathbf{u} - \mathbf{t})$ by a complex counterpart in (𝔼), the same derivation leads to

$$-iBp = \frac{d^2(\ln I)}{ds^2}\,,$$

where the condition $\mu(\mathbf{u}, t_1, \ldots, t_n)e^{-ipt_i^2/2} \to_{t_i \to \infty} 0$ is satisfied by requiring that μ is regular enough and vanishes at infinity, i.e.: $\mu(\mathbf{u}, t_1, \ldots, t_n) \to_{t_i \to \infty} 0$.

3.2. CONNECTION WITH THE FORMALISM OF QUANTUM MECHANICS

Setting $p' = 1/2x'^0$, we recognize that if μ does not depend on p, the product

$$\Psi = (2\sqrt{\pi x'^0}\,)^{-n} \cdot I$$

is a generic solution of the equation of the heat (x'^0 varies as the time variable t, and Ψ represents the temperature) [10]:

$$\frac{\partial \Psi}{\partial x'^0} - \Delta\Psi = 0\,,$$

where Δ is the Laplace operator: $\Delta = \nabla^2 = \partial^2/(\partial x^1)^2 + \ldots + \partial^2/(\partial x^n)^2$. Formally, if x'^0 is no longer a real number but a pure imaginary number ($x'^0 = ix^0$, x^0 real), then, this equation is a Schrödinger-type equation:

$$i\frac{\partial \Psi}{\partial x^0} = -\Delta\Psi\,,$$

where the Hamiltonian reduces to the Laplacian kinetic term: no potential term takes place. However, in the preceding treatment, p and B are considered as constant parameters. The function μ is therefore allowed to depend on p. Suppose that

μ has the form: $\mu(\mathbf{y}) = \alpha(p)\beta(\mathbf{y})$ where $\alpha(p)$ does not depend on $\mathbf{y}$ and where $\beta(\mathbf{y})$ does not depend on p. Then, Ψ/α is still a generic solution of the above Schrödinger-type equation for $p = -i/2x^0$, and therefore Ψ is a generic solution of a Schrödinger-type equation with a uniform term:

$$i\frac{\partial}{\partial x^0}\left(\frac{\Psi}{\alpha}\right) = -\Delta\left(\frac{\Psi}{\alpha}\right), \quad \Psi(x^0 = 0, x, y, z) = \alpha(p = i\infty)\mu(x, y, z),$$

$$i\frac{\partial\Psi}{\partial x^0} = -\Delta\Psi + V(x^0)\Psi = H\Psi, \quad \text{with} \quad V(x^0) = i\frac{d(\ln\alpha)}{dx^0}.$$

In quantum mechanics, a wave function accross the whole *space* is associated with a particle (or a system of particles) which is endowed with a fixed set S of extensive parameters (mass, charge, spin, etc.) and which is subjected to an external intensive potential P. S and P give rise to a potential energy $\mathbf{V}$ of the particle. Since the space is defined by its filling (e.g. the vacuum as a borderline case), a wave function might, in turn, be associated with the space itself. Such a wave function would be defined by a Schrödinger-type equation.

In order to interpret the two previous examples as borderline cases of a more general interpretation, it can be naturally suggested that the quantity

$$\mathbf{V}(x^0, \mathbf{u}) = \frac{i\dfrac{\partial\Psi}{\partial x^0} + \Delta\Psi}{\Psi}$$

represents some kind of complex "*potential energy of the space*". In other words, Ψ is a solution of the Schrödinger-type equation $\mathbf{H}\Psi + i\partial\Psi/\partial x^0 = 0$, where $\mathbf{H} = \mathbf{T} + \mathbf{V}$ is a Hamiltonian operator *with a complex "potential energy" term* (which now depends on both time- and space-coordinates). The function μ essentially determines this potential energy in that sense that $\mathbf{V}(p, \mathbf{u}) = 0$ if μ does not depend on p and that $\mathbf{V}(p, \mathbf{u})$ is uniform (i.e. only "time-dependent") if μ has the form $\mu(\mathbf{y}) = \alpha(p)\beta(\mathbf{y})$.

3.3. CONNECTION WITH THE FORMALISM OF GENERAL RELATIVITY

General relativity states that the space-time is a Riemannian manifold endowed with metric $g_{ij}dx^i dx^j$ ($i = 0, 1, 2, 3$) which is determined by the mass-energy flow entering the Einstein equations. The datum of a "spatial wave function" in the (non-physical) equation $-Bpds^2 = d^2 \ln \Psi$ defines a "complex metric" ds^2, the real part of which might be identified with the spatial part of a space-time metric at each time-like parameter $p = -i/2x^0$. However, this speculative analysis does not give a complete space-time-like metric, for the variable "time" is not represented in the vector $\mathbf{u}$ ($\mathbf{u}$ is not a 4-vector).

Direct problem. We are given some affine scalar of a real Euclidean space T_n, which is to be expressed in some rectangular coordinate system $\{x^1, \ldots, x^n\}$ by: μ:

$R^n \to R$ (or C). The expressions $\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(x)$ and $K_p(\mathbf{u}, \mathbf{u}+d\mathbf{u})$ are then written down for the group of translation of T_n, at each point $\mathbf{u}$ marked by $(x^1, \ldots, x^n)$ in R^n: eq. ($\mathbb{E}$) is then solved, and a linear element ds^2 is brought up. The pair (R^n, ds^2) is interpreted as one description of a Riemannian manifold X_n in the same way as (R^n, ds_e^2) is a description of the affine Euclidean space T_n (where $ds_e^2 = (\mathrm{d}x^1)^2 + \ldots + (\mathrm{d}x^n)^2$).

Equation ($\mathbb{E}$) would play two roles:

(a) it introduces a supplementary "time coordinate", p;

(b) it transforms the flat space T_n into a distorted (curved) space X_n.

Equation ($\mathbb{E}$) can be regarded as an application of a "time" variable onto an affine space, where μ plays the role of an "*initial datum*". A translation $\mathbf{t}$ makes a connection between two points of T_n. A component t_i operates independently on each direction "i", like a "time potential" which generates the possible motion $x^i \to x^i + t_i$ in T_n along the direction "i". The "*time coordinate*" $x^0 = 1/(2p)$ would then be defined from the action of the group of translations in R^3 and the corresponding equation ($\mathbb{E}$).

Converse problem. We are now given a space-time metric $g_{ij}dx^i dx^j$ $(i = 0, 1, 2, 3)$ issued from the Einstein equations, the pure spatial part being denoted $dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta$ $(0 \neq \alpha, \beta = 1, 2 \text{ or } 3$: $\gamma_{\alpha\beta} = -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00})$ [11]. We search for a corresponding wave function Ψ (or its "initial datum" μ), and more precisely, for one coordinate system $\{x^i\}$ of X_n satisfying:

(a) the open set of R^4 covered by the x^i's is R^4 in its entirety;

(b) the expression of dl^2 in this coordinate system is a solution of some equation ($\mathbb{E}$) written down for the group of translations in R^3 and for a function μ of the x^α's. From the preceding section, it follows that this condition is equivalent to the existence of a function $A(x^0, x^1, x^2, x^3)$ such that: $dl^2 = d_s^2 A$, where d_s^2 refers to the three space coordinates only, i.e.

$$\gamma_{\alpha\beta} = \frac{\partial^2 A}{\partial x^\alpha \partial x^\beta} \quad \alpha, \beta = 1, 2, 3 \text{ (six terms)}.$$

Example: search for space wave functions of Robertson and Walker spaces

It must be henceforth stressed that the convolution form for Ψ does not allow for describing the simplest non-flat space in rectangular coordinates. Robertson and Walker space-times are compared with a completely isotropic perfect fluid. Indeed, the metric of such spaces is given in spherical coordinate, by

$$ds_{rw}^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin\theta\, d\phi^2)\right]$$

(here: $x^0 = ct$ or ict, $k = 0$, 1 or -1, and the Riemannian curvature equals $k/R^2(t)$).

Since $dx^2 + dy^2 + dz^2 = dr^2 + r^2d\theta^2 + r^2\sin\theta\, d\phi^2$, it can be shown that in Cartesian coordinates

$$ds_{rw}^2 = c^2dt^2 - R^2(t)\left[\frac{1-k(y^2+z^2)}{1-kr^2}dx^2 + \frac{1-k(x^2+z^2)}{1-kr^2}dy^2 + \frac{1-k(y^2+x^2)}{1-kr^2}dz^2 + \frac{2kxy}{1-kr^2}dxdy + \frac{2kxz}{1-kr^2}dxdz + \frac{2kzy}{1-kr^2}dzdy\right],$$

where $r^2 = r^2(x,y,z) = x^2 + y^2 + z^2$, $k = 0$, $+1$ or -1, and where R is a positive function of time. Notice that both the spherical and Cartesian coordinate systems correspond to synchronous referentials, i.e. they satisfy: $g_{0\alpha} = 0$ for $\alpha \neq 0$, and consequently: $g_{\alpha\beta} = -\gamma_{\alpha\beta}$. The spatial time-dependent tridimensional metric reads: $dl_{rw}^2 = -ds_{rw}^2 + c^2dt^2$, i.e.

$$dl_{rw}^2 = R^2(t)\left[\frac{1-k(y^2+z^2)}{1-kr^2}dx^2 + \frac{1-k(x^2+z^2)}{1-kr^2}dy^2 + \frac{1-k(y^2+x^2)}{1-kr^2}dz^2 + \frac{2kxy}{1-kr^2}dxdy + \frac{2kxz}{1-kr^2}dxdz + \frac{2kzy}{1-kr^2}dzdy\right].$$

- $k = -1$ *or* $+1$ ($k \neq 0$, V_4 is non-flat)

Let us assume that $ds^2 = dl_{rw}^2$. Then, it is necessary that there exists a function $A_p(x,y,z)$ such that: $dl^2 = d^2A$. Given such a function A_p, since $-Bpdl^2 = d^2(\ln\Psi)$, we would have to seek for a function I satisfying

$$A_p = -\frac{1}{Bp}\ln[(2\sqrt{\pi i x^0})^{-n}\cdot I/\alpha(p)],$$

i.e. a function Ψ such that

$$\ln\Psi(p,x,y,z) = -BpA_p(x,y,z)) + \ln\alpha(p),$$

where the term $\alpha(p)$ is actually unessential (it has been supposed that p varies with time only). In particular,

$$\frac{\partial^2 A_p}{\partial x\partial y} = R^2(t)\frac{kxy}{1-kr^2} \Rightarrow \frac{\partial(A_p/R^2)}{\partial x} = -\frac{x}{2}\ln(1-kr^2) + h_1(x,z)$$

and likewise,

$$\frac{\partial^2(A_p/R^2)}{\partial x\partial z} = \frac{kxz}{1-kr^2} \Rightarrow \frac{\partial(A_p/R^2)}{\partial x} = -\frac{x}{2}\ln(1-kr^2) + h_2(x,y),$$

where h_1 and h_2 are any differentiable functions, respectively independent of y and z. However, equating the two former expressions yields

$$h_1(x,z) = h_2(x,y) = h(x)\,, \quad \text{where } h \text{ is independent of } y \text{ and } z\,.$$

Therefore, the partial derivative of $\partial(A_p/R^2)/\partial x$ with respect to x gives

$$\frac{\partial^2(A_p/R^2)}{\partial x^2} = -\frac{1}{2}\ln(1-kr^2) + \frac{x}{2}\,\frac{2kx}{1-kr^2} + h'(x)\,.$$

On the other hand, $dl^2 = d^2A_p$ also entails

$$\frac{\partial^2(A_p/R^2)}{\partial x^2} = \frac{1-k(y^2+z^2)}{1-kr^2}\,.$$

Thus, $h'(x) = 1 + \frac{1}{2}\ln(1-kr^2)$. Obviously, the term on right-hand side depends on y and z ($k \neq 0$) while $h'(x)$ does not. In conclusion, the selected rectangular coordinate system does not allow for a description of hyperbolic or spheric Robertson and Walker spaces by means of any spatial wave functions Ψ as defined above.

- $k = 0$ (V_4 is flat)

Then, $dl^2 = R^2(t)[dx^2 + dy^2 + dz^2]$, and the function $A(x,y,z) = A_p(x,y,z) = [R^2(t)/2](x^2+y^2+z^2)$ fulfills the required condition $dl^2 = d^2A$. The subsequent equation $\Psi = \exp[-pBA + \ln\alpha(p)]$ is equivalent to the search for a density μ such that

$$(2\sqrt{\pi i x^0}\,)^{-3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mu(x-t_1,y-t_2,z-t_3)e^{-p(t_1^2+t_2^2+t_3^2)/2}\,dt_1\,dt_2\,dt_3$$
$$= \alpha(p)\exp\left[-\frac{BpR^2(t)}{2}(x^2+y^2+z^2)\right],$$

where $x^0 = 1/(2ip)$ and where μ eventually depends on p. In order to get a solution μ of the form: $\mu(x,y,z) = m_p(x)m_p(y)m_p(z)$, we seek for a real or complex function m_p such that

$$\int_{-\infty}^{+\infty} m_p(x-t)e^{-pt^2/2}\,dt = (2\sqrt{\pi i x_0}\,)\alpha^{1/3}\exp\left[-\frac{pBR^2}{2}x^2\right].$$

This is written as

$$\int_{-\infty}^{+\infty} m_p(t)e^{-p(x^2+t^2-2xt)/2}\,dt = \sqrt{\frac{2\pi}{p}}\,\alpha^{1/3}\exp\left[-\frac{pBR^2}{2}x^2\right]$$

or

$$\int_{-\infty}^{+\infty} m_p(t)e^{-pt^2/2}e^{pxt}\,dt = \sqrt{\frac{2\pi}{p}}\,\alpha^{1/3}\exp\left[\frac{p(1-BR^2)}{2}x^2\right].$$

After the variable change $t \rightarrow u = -pt$, we get

$$\frac{1}{p}\int_{-\infty}^{+\infty} m_p(-u/p)e^{-u^2/(2p)}e^{-ux}\,du = \sqrt{\frac{2\pi}{p}}\,\alpha^{1/3}\exp\left[\frac{p(1-BR^2)}{2}x^2\right]$$

i.e.

$$\mathcal{L}(f_p)(x) = \sqrt{2\pi p}\,\alpha^{1/3}\exp\left[\frac{p(1-BR^2)}{2}x^2\right], \tag{I}$$

where $\mathcal{L}(f_p)$ denotes the Laplace transform of the continuously derivable function $f_p(u) = m_p(-u/p)e^{-u^2/(2p)}$. Conversely, $f_p(u)$ is given by the formula of Mellin–Fourier:

$$f_p(u) = \frac{1}{2i\pi}\int_{-i\infty}^{+i\infty} \mathcal{L}(f_p)(z)e^{zu}\,dz\,.$$

The function

$$z \to \exp\left[\frac{p(1-BR^2)}{2}z^2\right]$$

is continuously holomorph on $\mathbb{C}$ and coincides with

$$x \to \exp\left[\frac{p(1-BR^2)}{2}x^2\right]$$

on $\mathbb{R}$. Thus,

$$f_p(u) = \sqrt{2\pi p}\,\frac{\alpha^{1/3}}{2i\pi}\int_{-i\infty}^{+i\infty}\exp\left[\frac{p(1-BR^2)}{2}z^2\right]e^{zu}\,dz\,,$$

i.e. $(z = iv)$

$$\begin{aligned} f_p(u) &= \sqrt{2\pi p}\,\frac{\alpha^{1/3}}{2\pi}\int_{-\infty}^{+\infty}\exp\left[\frac{p(1-BR^2)}{2}(-v^2)\right]e^{ivu}\,dv \\ &= \sqrt{\frac{p}{2\pi}}\,\alpha^{1/3}2\int_{0}^{+\infty}\exp\left[-\frac{p(1-BR^2)}{2}v^2\right]\cos(vu)\,dv \\ &= \sqrt{\frac{2p}{\pi}}\,\alpha^{1/3}\frac{\sqrt{\pi}}{2\sqrt{\frac{p(1-BR^2)}{2}}}\exp\left[\frac{-u^2}{4\frac{p(1-BR^2)}{2}}\right]. \end{aligned}$$

And finally,

$$m_p(-u/p) = \frac{\alpha^{1/3}}{\sqrt{1-BR^2}}\exp\left[\frac{-BR^2u^2}{2p(1-BR^2)}\right],$$

i.e. $(x = -u/p)$

$$m_p(x) = \frac{\alpha^{1/3}}{\sqrt{1 - BR^2}} \exp\left[\frac{-BpR^2}{2(1 - BR^2)} x^2\right].$$

In conclusion, the density

$$\begin{aligned}\mu(p, x, y, z) &= m_p(x) m_p(y) m_p(z) \\ &= \alpha(p) \left[\frac{1}{1 - BR^2}\right]^{3/2} \exp\left[\frac{-BpR^2}{2(1 - BR^2)} (x^2 + y^2 + z^2)\right]\end{aligned}$$

gives rise to

$$\ln \Psi = \frac{-BpR^2(t)}{2} (x^2 + y^2 + z^2) + \ln \alpha(p).$$

Prior to the calculation of the constant B, let us come back to the speculative interpretation of Ψ.

- *Condition for a uniform "potential energy" term*

It should be emphasized that, in general, the potential energy of the space is not uniform, for μ does not read $\alpha(p)\beta(x, y, z)$, where β would not depend on p. However, the latter condition is fulfilled as soon as the term B satisfies the equation

$$\frac{d}{dp}\left\{\frac{BpR^2}{1 - BR^2}\right\} = 0,$$

i.e. $BpR^2/(1 - BR^2) = p_0$, *constant* with respect to the variables x, y, z *and* p (*or* t, *or* x^0).

Thus, the "potential energy" of the space is uniform only if B is subjected to vary with p as

$$B = \frac{1}{1 + p/p_0} \frac{1}{R^2}.$$

Since p is proportional to the reciprocal of the "time", then

$$B = \frac{1}{1 + t_0/t} \frac{1}{R^2},$$

where t_0 corresponds to p_0. If t_0 is interpreted as an "initial time" and $t_0 = 0$ (i.e. $p_0 = \infty$), then,

$$B = \frac{1}{R^2(t)} \quad \text{and} \quad q = \infty.$$

Under the above condition,

$$\ln \Psi = \frac{-p}{2} (x^2 + y^2 + z^2) + \ln \alpha(p)$$

or

$$\Psi = \alpha(p) \exp\left[\frac{-pr^2}{2}\right].$$

- *Derivation of the constant term B*

  B is to be calculated from the postulated relationship

$$B = q^{-3} \frac{\left(\int_{R^3} \mu^2(x,y,z)\, dx\, dy\, dz\right)^2}{\left(\int_{R^3} \mu(x,y,z)\, dx\, dy\, dz\right)^4} = q^{-3} \frac{\left(\int_{-\infty}^{+\infty} \exp\left[\frac{-BpR^2}{1-BR^2}x^2\right] dx\right)^6}{\left(\int_{-\infty}^{+\infty} \exp\left[\frac{-BpR^2}{2(1-BR^2)}x^2\right] dx\right)^{12}}.$$

By using the known result $\int_{-\infty}^{+\infty} e^{-a^2x^2}\, dx = \sqrt{\pi}/a$ for $a^2 = bpR^2/1 - BR^2$ and $a^2 = \frac{1}{2}BpR^2/1 - BR^2$, we get

$$B = q^{-3}\left[\frac{BpR^2}{\pi(1-BR^2)}\right]^3.$$

In a direct interpretation of the early equation ($\mathbb{E}$), q is the volume of R^3: replacing q by ∞ in the above relationship, we are lead to the equation $BR^2 = 1$.

Therefore, *the very first formulation of eq.* ($\mathbb{E}$) *infers that the metric of flat Robertson and Walker spaces correspond to a Schrödinger-type equation with a* ***uniform*** *"potential energy" term.*

- *Calculation of the wave function of flat Robertson and Walker spaces*

  With the condition $BR^2(t) = 1$, eq. (I) reads $\mathcal{L}(f_p)(x) = \sqrt{2\pi p}\,\alpha^{1/3}$. It entails

$$f_p(x) = \delta(x)F_p(x),$$

where δ denotes the Dirac distribution, and where $F_p(x)$ is a function satisfying $F_p(0) = \sqrt{2\pi p}\,\alpha^{1/3}$. Thus, $m_p(x) = \delta(-px)F_p(-px) \cdot e^{px^2/2}$, and consequently,

$$\begin{aligned}\mu(p,x,y,z) &= m_p(x)m_p(y)m_p(z)\\ &= \delta(-px)\delta(-py)\delta(-pz)F_p(-px)F_p(-py)F_p(-pz) \cdot e^{p(x^2+y^2+z^2)/2}.\end{aligned}$$

The condition $BR^2(t) = 1$ would define Ψ, but not the "mother function" μ, unless μ is a distribution. But then

$$B = q^{-3} \frac{\left(\int_{R^3} \mu^2(x,y,z)\, dx\, dy\, dz\right)^2}{\left(\int_{R^3} \mu(x,y,z)\, dx\, dy\, dz\right)^4} = q^{-3}\frac{(\delta(0)F_p^2(0))^6}{(F_p(0))^{12}}$$

$$= q^{-3}\delta^3(0) = \infty \quad \text{unless} \quad q = a\delta(0).$$

As $BR^2 = 1$ and $R^2(t) \neq 0$, then $B \neq \infty$ and it is confirmed that $q = \infty$.

The "potential energy" term is calculated from the definition

$$\mathbf{V}(x^0, \mathbf{u}) = \frac{i\dfrac{\partial\Psi}{\partial x^0} + \Delta\Psi}{\Psi} = i\left[\frac{\partial \ln \alpha(p)}{\partial x^0} + \frac{3}{2x^0}\right] .$$

Remark

The expression "$V(x^0) = i\mathrm{d}(\ln\alpha)/\mathrm{d}x^0$" derived in section 3.2 for uniform "potential energy" terms does not apply with the present definition of $\alpha(p)$. The value of $\mathbf{V}$ could also be derived by putting $(2\pi p)^{3/2}\alpha(p)$ in place of α in the direct expression of $V(x^0)$.

It is noteworthy that the "potential energy" term does not vanish, even if no variation with time is introduced *a priori*, i.e. if $\alpha(p)$ is constant:

$$\Psi = \kappa \exp\left[\frac{-pr^2}{2}\right] \Rightarrow \mathbf{V}(x^0) = \frac{3i}{2x^0} .$$

4. Conclusion

It cannot be overemphasized that the application of chemical algebra to the background of mathematical physics is purely speculative. In particular, the non-tensorial character of eq. ($\mathbb{E}$) does not receive a straightforward interpretation. Moreover, the last speculations would be ambiguous as time and space variables are not treated in a homogeneous manner: the statement that no space exists without a time and vice versa is reflected in the definition of quadridimensional space-time. From an *axiomatic viewpoint*, both the time variable and the space variables cannot be deduced from each other, and the whole quadridimensional space-time is to be introduced at the outset. The consequences of this principle will be developed within the framework of chemical algebra.

References and notes

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[3] R. Chauvin, Paper III of this series, J. Math. Chem. 16 (1994) 269.
[4] R. Chauvin, Paper IV of this series, J. Math. Chem. 16 (1994) 285.
[5] J.P. Serre, *Représentations Linéaires des Groupes Finis*, 3rd ed. (Hermann, Paris, 1978).
[6] R. Chauvin, Paper V of this series, J. Math. Chem. 17 (1995) 235.
[7] On the very outset, we may examine some particular general assumptions on μ which are different from that developped in the text (hypothesis 3).

- *Hypothesis 1:* $\forall \mathrm{t} \in G, \mu(\mathrm{t}) = \mu(-\mathrm{t})$

Then it is easily checked that $J_i = 0$. Thus,

$$B(\mathbf{u})ds^2 = -\frac{1}{p}\sum_{i=1}^{n}\frac{\displaystyle\int \frac{\partial^2\mu}{\partial t_i \partial t_j} e^{-p\|\mathbf{t}\|^2/2}\, d\tau}{I}\, dx_i^2 \, .$$

Non-Euclidean distances with non-constant coefficients $g_{ii}(\mathbf{u})$ occur only if μ depends on $\mathbf{u}$.

• *Hypothesis 2:* $\mu(\mathbf{u},\mathbf{t}) = \mu_1(\mathbf{u},t_1)\ldots\mu_n(\mathbf{u},t_n)$

The multiple integrals reduce to simple integrals.

$$I = \left\{\int_{-\infty}^{+\infty} \mu_1(\mathbf{u},t)e^{-pt^2/2}\, dt\right\}\ldots\left\{\int_{-\infty}^{+\infty} \mu_n(\mathbf{u},t)e^{-pt^2/2}\, dt\right\} = I_1 \ldots I_n \, ,$$

$$\int \frac{\partial\mu}{\partial t_i} e^{-p\|\mathbf{t}\|^2/2}\, d\tau = \int_{-\infty}^{+\infty} \frac{\partial\mu_i}{\partial t}(\mathbf{u},t)e^{-pt^2/2}\, dt \cdot \frac{I}{I_i} \, ,$$

$$\int \frac{\partial^2\mu}{\partial t_i \partial t_j} e^{-p\|\mathbf{t}\|^2/2}\, d\tau = \int_{-\infty}^{+\infty} \frac{\partial\mu_i}{\partial t}(\mathbf{u},t)e^{-pt^2/2}\, dt \int_{-\infty}^{+\infty} \frac{\partial\mu_j}{\partial t}(\mathbf{u},t)e^{-pt^2/2}\, dt \cdot \frac{I}{I_i I_j} \quad (i \neq j) \, ,$$

$$\int \frac{\partial^2\mu}{\partial t_i^2} e^{-p\|\mathbf{t}\|^2/2}\, d\tau = \int_{-\infty}^{+\infty} \frac{\partial^2\mu_i}{\partial t^2}(\mathbf{u},t)e^{-pt^2/2}\, dt \cdot \frac{I}{I_i} \, .$$

Therefore, the coefficients of the crossed terms $dx_i dx_j$, $i \neq j$, are nul. A reduced form of the metric is thus obtained in Cartesian coordinates:

$$B(\mathbf{u})ds^2 = \frac{1}{p}\sum_{i=1}^{n}\left(\frac{1}{I_i^2}\left\{\int_{-\infty}^{+\infty} \frac{\partial\mu_i}{\partial t}(\mathbf{u},t)e^{-pt^2/2}\, dt\right\}^2 - \frac{1}{I_i}\int_{-\infty}^{+\infty} \frac{\partial^2\mu_i}{\partial t^2}(\mathbf{u},t)e^{-pt^2/2}\, dt\right) dx_i^2 \, .$$

• *Hypothesis 3:* I, J_i, L_{ij} are convolution products. This hypothesis meets the framework detailed in text.

[8] D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles* (Dover, New York, 1989).

[9] Suppose that the left member of ($\mathbb{E}$) under its differential form still reduces to: $\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(\gamma ds) - 1 = \mathrm{p}ds^2$. If ds^2 means a (positive or negative) squared distance, and if p is a real parameter, the local pairing product $K_p(\mathbf{u},\mathbf{u}+d\mathbf{u})$ on the right-hand side of the equation must be a real number, even if μ is complex. It could be therefore suggested that in such a case, the upper and the lower products in K are replaced by two scalar products between members of the $\mathbb{R}$-vector space $\mathbb{C}$ identified to $\mathbb{R}^2$. However, if p is an imaginary number ($p = ip'$, $p' \in \mathbb{R}$), the left-hand side of ($\mathbb{E}$) is a pure imaginary number $\Phi_{\mathbf{u},\mathbf{u}+d\mathbf{u}}(\gamma ds) - 1 = ip'ds^2$, but there is no natural way to reduce the right-hand side $K_{ip'}(\mathbf{u},\mathbf{u}+d\mathbf{u})$ to a pure imaginary number under such a condition.

[10] A. Guichardet, *Calcul Intégral* (Libraire Arman Colin, Paris, 1969) pp. 196–203.

[11] L. Landau and E. Lifchitz, Théorie des champs, *Physique Théorique*, Vol. 2, 4th French Ed. (Mir, Moscow, 1989).